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Parabolic Equation Starting Field For a Prolate Spheroid Source

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Preface

This report was prepared while one of the authors, Dr. G. R. Verma of the University of Rhode Island, was on an Intergovernmental Personnel Act Mobility Assignment at the Naval Underwater Systems Center.

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Parabolic equations are used to calculate the propagation of sound over distance. To use parabolic equations, one must be able to specify the starting field. At close ranges, the acoustic source should not be considered as a point source. ~~We derive~~ the starting field are derived for the special case of a prolate spheroid source in an infinite uniform half-space. Keywords:

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INTRODUCTION

Parabolic equations are used to calculate the propagation of acoustic fields over long distances in waveguides. To use parabolic equations one must specify initial values. In the case of sound propagation in the ocean, the inital values required are the so-called "starting field" which must be given from the surface to the bottom of the ocean completely surounding the source of sound. Once this is given, a numerical parabolic equation model can be used to advance this field one horizontal step at a time to any desired range. We are interested in calculating such fields at rather close ranges. These short ranges imply that the source should not be considered to be a point source. For this paper we model very special sources as vibrating prolate spheroids, such sources being cigar shaped. We examine this special problem to give an idea of the effects on the starting field, and hence on the acoustic field when the source is not a point source.

SEPARATION IN PROLATE SPHEROIDAL COORDINATES

The sound field in a uniform medium is governed by the wave equation given by

$$\nabla^2\psi = \frac{1}{c^2}\psi_{tt}. \tag{1}$$

We want to solve this equation in a uniform halfspace that contains a vibrating prolate spheroid. We follow the usual approach of eigenvalue expansion [ref. 1-3]. Instead of the normal Euclidean coordinate system (x, y, z) a prolate spheroidal coordinate system (μ, θ, ϕ) is used where the two systems are related by

$$x = a/2 \sinh(\mu)\sin(\theta)\cos(\phi), \tag{2}$$

$$y = a/2 \sinh(\mu)\sin(\theta)\sin(\phi) \text{ and} \tag{3}$$

$$z = a/2 \cosh(\mu)\cos(\theta). \tag{4}$$

If we write $\cosh(\mu) = \xi$ and $\cos(\theta) = \eta$, then we get

$$x = \frac{a}{2}\sqrt{\xi^2 - 1}\sqrt{1 - \eta^2}\cos(\phi), \tag{5}$$

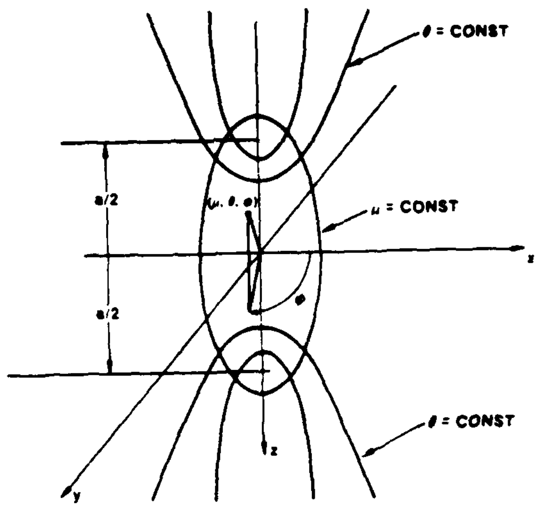
$$y = \frac{a}{2}\sqrt{\xi^2 - 1}\ \sqrt{1 - \eta^2}\ \sin(\phi) \text{ and} \tag{6}$$

$$z = \frac{a}{2}\xi\eta, \tag{7}$$

where the ranges of ξ, η and ϕ are given by $1 \le \xi \le \infty$, $-1 \le \eta \le 1$ and $0 \le \phi < 2\pi$. Prolate spheroids are defined by μ = constant or ξ = constant; whereas θ= constant or η = constant give hyperboloids of two sheets and ϕ = constant gives the half planes through the z-axis (See Figure 1).

Figure 1

PROLATE SPHEROIDAL RADIATOR



If we substitute

$$\psi = S(\eta)R(\xi)\Phi(\phi)e^{-iwt} \tag{8}$$

into equation (1) we get the ordinary differential equations

$$\frac{d}{d\eta}\left[(1 - \eta^2)\frac{dS}{d\eta}\right] = -\left[A - h^2\eta^2 - \frac{m^2}{1 - \eta^2}\right]S\ , \tag{9}$$

$$\frac{d}{d\xi}\left[(\xi^2 - 1)\frac{dR}{d\xi}\right] = \left[A - h^2\xi^2 + \frac{m^2}{\xi^2 - 1}\right]R \tag{10}$$

and

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi\ , \tag{11}$$

where A and m are separation constants, and $h = ka/2$ where $k = \omega/c$.

As $h \to 0$, the spheroidal coordinates go to spherical co-ordinates and the S-functions go to associated Legendre polynomials $P_\ell^m(\eta)$. When $h \neq 0$ we denote the dependence of S-functions on ℓ, m and h and write them as

$$S(\eta) = S(h,\eta,m,\ell) = \sum_{n=0}^{\infty} d(h,m,\ell,2n)P^m_{m+2n}(\eta), \quad \text{when } m-\ell \text{ is 0 or even} \tag{12}$$

or

$$S(\eta) = S(h,\eta,m,\ell) = \sum_{n=0}^{\infty} d(h,m,\ell,2n+1)P^m_{m+2n+1}(\eta) \quad \text{when } m-\ell \text{ is odd .} \tag{13}$$

The standard notation in the literature [refs. 1,2,4] is given as $S(\eta) = \Sigma' \, d(h,m,\ell,n)P^m_{m+n}(\eta)$ where the ' denotes the proper choice of (12) or (13) depending on the parity of $(m-\ell)$.

The radial solutions $R(\xi)$ show a structure similar to the angular functions $S(\eta)$. The two solutions $R^{(1)}$ and $R^{(2)}$ can be expressed in terms of spherical Bessel functions as follows

$$R^{(1)}(h\xi,m,\ell) = \left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} \sum_{n=0}^{\infty}{}' \, a(h,m,\ell,n)j_{m+n}(h\xi) \tag{14}$$

and

$$R^{(2)}(h\xi,m,\ell) = \left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} \sum_{n=0}^{\infty}{}' \, a(h,m,\ell,n)n_{m+n}(h\xi) \; . \tag{15}$$

For expressing complex radial solutions propagating outward from the prolate spheroid in terms of spherical Hankel functions we use

$$R^{(3)} = R^{(1)} + iR^{(2)}, \tag{16}$$

because it satisfies the Sommerfeld radiation condition.

A VIBRATING PROLATE SPHEROIDAL SOURCE

We consider a prolate spheroid whose surface $\xi = \xi_0$ vibrating in its fundamental mode with a velocity

$$v = S(h,\eta,0,0)(\xi_0^2 - \eta^2)^{-1/2} \, e^{-i\omega t}. \tag{17}$$

We seek the corresponding pressure field in the form

$$p = A \; S(h,\eta,0,0) \; R(h\xi,0,0) \; e^{-i\omega t} \; , \tag{18}$$

Where A is a constant. The balance of force, mass and acceleration is [ref. 6]

$$\nabla p = -\rho_0 \frac{\partial \vec{v}}{\partial t} , \tag{19}$$

which expanded in vector form is [ref. 6]

$$\left[\frac{1}{\sqrt{\xi_0^2 - \eta^2}} \frac{\partial p}{\partial \eta} , \frac{1}{\sqrt{\xi_0^2 - \eta^2}} \frac{\partial p}{\partial \xi} , 0\right] = \left[0, \frac{i\omega\rho^0}{\sqrt{\xi_0^2 - \eta^2}} S(h,\eta,0,0)e^{-i\omega t},0\right] \tag{20}$$

The first two components evaluated at the surface of the prolate spheroid give

$$\frac{1}{\sqrt{\xi_0^2 - \eta^2}} A\ S'(h,\eta,0,0)R(h\xi_0,0,0)e^{-i\omega t} = 0 \tag{21}$$

and

$$\frac{hA}{\sqrt{\xi_0^2 - \eta^2}} S(h,\eta,0,0)R'(h\xi_0,0,0)e^{-i\omega t} = \frac{i\omega\rho^0}{\sqrt{\xi_0^2 - \eta^2}} S(h,\eta,0,0)e^{-i\omega t}. \tag{22}$$

Equation (21) states that ξ_0 must be a zero of $R(h\xi,0,0)$, that is to say, ξ_0 must be an eigenvalue. Equation (22) is the equation from which we derive the value of A

$$A = \frac{-ia\omega}{2h} \frac{-\rho_0}{R^{(3)'}(h\xi_0,0,0)} . \tag{23}$$

The pressure p is therefore expressed as

$$p = \frac{ia\omega\rho_0}{2h} \frac{R^{(3)}(h\xi,0,0)}{R^{(3)'}(h\xi_0,0,0)} S(h,\eta,0,0)\ e^{-i\omega t} \tag{24}$$

where ξ_0 is some eigenvalue of the vibration of the prolate spheroid.

TRANSFORMATION TO THE USUAL OCEAN COORDINATES

The prolate spheroid with its axis of symetry on the z-axis of a rectangular (x,y,z) coordinate system is transformed to a horizontal prolate spheroid in a rectangular (ρ,χ,ζ) coordinate system by the transformation (See Figure 2)

$$\rho = z , \tag{25}$$

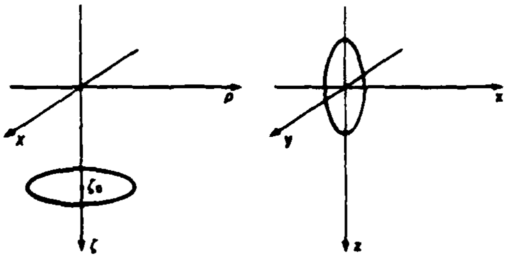
$$\chi = y \text{ and} \tag{26}$$

$$\zeta = x + \zeta_0 . \tag{27}$$

The transformation takes the center of the prolate spheroid in the (x,y,z) coordinate system to the point $(0,0,\zeta_0)$ in the (ρ,χ,ζ) coordinate system and the axis of symmetry is now parallel to the ρ axis.

Figure 2

TRANSFORMATION OF COORDINATES



If we solve the equations (2)-(4) for ξ and η we get

$$\xi = \frac{1}{a}\sqrt{x^2+y^2+(z+a/2)^2} + \frac{1}{a}\sqrt{x^2+y^2+(z-a/2)^2} \qquad (28)$$

and

$$\eta = \frac{1}{a}\sqrt{x^2+y^2+(z+a/2)^2} - \frac{1}{a}\sqrt{x^2+y^2+(z-a/2)^2} \; . \qquad (29)$$

When x, y and z are expressed in terms of ρ, χ and ζ we obtain

$$\xi = \frac{1}{a}\sqrt{(\zeta-\zeta_0)^2+\chi^2+(\rho+a/2)^2} + \frac{1}{a}\sqrt{(\zeta-\zeta_0)^2+\chi^2+(\rho-a/2)^2} \qquad (30)$$

and

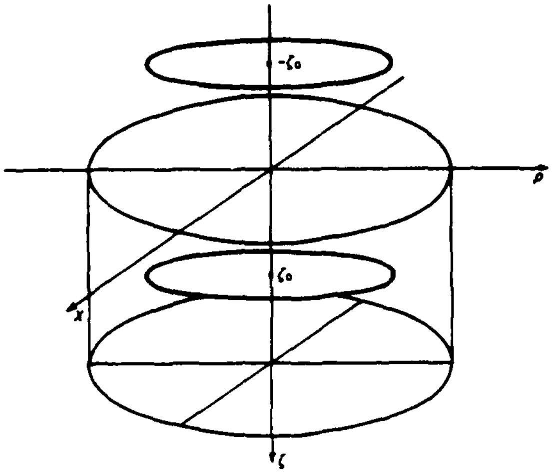
$$\eta = \frac{1}{a}\sqrt{(\zeta-\zeta_0)^2+\chi^2+(\rho+a/2)^2} - \frac{1}{a}\sqrt{(\zeta-\zeta_0)^2+\chi^2+(\rho-a/2)^2} \; . \qquad (31)$$

METHOD OF IMAGES

To generate our starting field we use the method of images involving two identical prolate spheroidal sources, opposite in phase; one at a depth of ζ_0 and another at $-\zeta_0$ (See Figure 3). This allows us to match a pressure release condition (p=0) at the surface of the ocean.

Figure 3

METHOD OF IMAGES



The total pressure is the superposition of the field from each individual source which is

$$p = \text{const } e^{i\omega t}[R^{(3)}(h\xi,0,0)S(h,\eta,0,0) - R^{(3)}(h\xi',0,0)S(h,\eta',0,0)] \tag{32}$$

where

$$\xi' = \frac{1}{a}\sqrt{(\zeta+\zeta_0)^2+\chi^2+(\rho+a/2)^2} + \frac{1}{a}\sqrt{(\zeta+\zeta_0)^2+\chi^2+(\rho-a/2)^2} \quad \text{and} \tag{33}$$

$$\eta' = \frac{1}{a}\sqrt{(\zeta+\zeta_0)^2+\chi^2+(\rho+a/2)^2} - \frac{1}{a}\sqrt{(\zeta+\zeta_0)^2+\chi^2+(\rho-a/2)^2} \;. \tag{34}$$

Finally, we obtain the starting field that corresponds to our special source, a prolate spheroid vibrating in its fundamental mode, by sampling the expression in (32) over the entire depth of the ocean at a fixed range. This starting field will vary with the direction from the source because the source is not rotationally symmetric about the ζ axis.

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